

On third Hankel determinants for subclasses of analytic functions and close-to-convex harmonic mappings

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Abstract

In this paper, we obtain the upper bounds to the third Hankel determinants for starlike functions of order α , convex functions of order α and bounded turning functions of order α . Furthermore, several relevant results on a new subclass of close-to-convex harmonic mappings are obtained. Connections of the results presented here to those that can be found in the literature are also discussed.

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1 Introduction

Let \mathcal{A} be the class of functions *analytic* in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1.1)$$

We denote by \mathcal{S} the subclass of \mathcal{A} consisting of univalent functions.

A function $f \in \mathcal{A}$ is said to be starlike of order α ($0 \leq \alpha < 1$), if it satisfies the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}).$$

We denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions of order α .

Denote by $\mathcal{K}(\alpha)$ the class of functions $f \in \mathcal{A}$ such that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (-1/2 \leq \alpha < 1; z \in \mathbb{D}).$$

In particular, functions in $\mathcal{K}(-1/2)$ are known to be close-to-convex but are not necessarily starlike in \mathbb{D} . For $0 \leq \alpha < 1$, functions in $\mathcal{K}(\alpha)$ are known to be convex of order α in \mathbb{D} .

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha)$, consisting of functions whose derivative have a positive real part of α ($0 \leq \alpha < 1$), if it satisfies the following condition:

$$\Re(f'(z)) > \alpha \quad (z \in \mathbb{D}).$$

Choosing $\alpha = 0$, we denote the $\mathcal{S} := \mathcal{S}^*(0)$, $\mathcal{K} := \mathcal{K}(0)$ and $\mathcal{R} := \mathcal{R}(0)$, the classes of starlike, convex and bounded turning functions, respectively.

Let \mathcal{H} denote the class of all *complex-valued harmonic mappings* f in \mathbb{D} normalized by the condition $f(0) = f_z(0) - 1 = 0$. It is well-known that such functions can be written as $f = h + \bar{g}$, where h and g are analytic functions in \mathbb{D} . We call h the analytic part and g the co-analytic part of f , respectively. Let \mathcal{S}_H be the subclass of \mathcal{H} consisting of univalent and sense-preserving mappings. Such mappings can be written in the form

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k} \quad (|b_1| < 1; \quad z \in \mathbb{D}). \quad (1.2)$$

Harmonic mapping f is called locally univalent and sense-preserving in \mathbb{D} if and only if $|h'(z)| > |g'(z)|$ holds for $z \in \mathbb{D}$. Observe that \mathcal{S}_H reduces to \mathcal{S} , the class of normalized univalent analytic functions, if the co-analytic part g vanishes. The family of all functions $f \in \mathcal{S}_H$ with the additional property that $f_{\bar{z}}(0) = 0$ is denoted by \mathcal{S}_H^0 . For further information about planar harmonic mappings, see e.g. [10, 13, 34].

Recall that a function $f \in \mathcal{H}$ is close-to-convex in \mathbb{D} if it is univalent and the range $f(\mathbb{D})$ is a close-to-convex domain, i.e., the complement of $f(\mathbb{D})$ can be written as the union of nonintersecting half-lines. A normalized analytic function f in \mathbb{D} is close-to-convex in \mathbb{D} if there exists a convex analytic function in \mathbb{D} , not necessarily normalized, ϕ such that $\Re(f'(z)/\phi'(z)) > 0$. In particular, if $\phi(z) = z$, then for any $f \in \mathcal{A}$, $\Re(f'(z)) > 0$ implies f is close-to-convex in \mathbb{D} , see [38]. We refer to [6, 20, 30, 35, 36] for discussion and basic results on close-to-convex harmonic mappings.

For a harmonic mapping $f = h + \bar{g}$ in \mathbb{D} , a basic result in [29] (see also [28]) shows that if at least one of the analytic functions h and g is convex, then f is univalent whenever it is locally univalent in \mathbb{D} . It is natural to study the univalence of $f = h + \bar{g}$ in \mathbb{D} if it is locally univalent and sense-preserving, and analytic function h is univalent and close-to-convex. Motivated by this idea, we next consider the following subclass of \mathcal{H} .

Definition 1. For $\alpha \in \mathbb{C}$ with $-1/2 \leq \alpha < 1$, let $\mathcal{M}(\alpha)$ denote the class of harmonic mapping f in \mathbb{D} of the form (1.2), with $h'(0) \neq 0$, which satisfy

$$\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > \alpha \quad \text{and} \quad g'(z) = zh'(z) \quad (z \in \mathbb{D}).$$

By making use of the similar arguments to those in the proof of [7, Theorem 1], one can easily obtain the close-to-convexity of the class $\mathcal{M}(\alpha)$. For special values of α , many authors have studied the class of close-to-convex harmonic mappings, see e.g. [5, 9, 29, 30, 39].

Pommerenke (see [32, 33]) defined the Hankel determinant $H_{q,n}(f)$ as

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \quad (q, n \in \mathbb{N}).$$

Problems involving Hankel determinants $H_{q,n}(f)$ in geometric function theory originate from the work of, e.g., Hadamard, Polya and Edrei (see [11,14]), who used them in study of singularities of meromorphic functions. For example, they can be used in showing that a function of bounded characteristic in \mathbb{D} , i.e., a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational [8]. Pommerenke [32] proved that the Hankel determinants of univalent functions satisfy the inequality $|H_{q,n}(f)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$, where $\beta > 1/4000$ and K depends only on q . Furthermore, Hayman [17] has proved a stronger result for areally mean univalent functions, i.e., the estimate $H_{2,n}(f) < An^{1/2}$, where A is an absolute constant.

We note that $H_{2,1}(f)$ is the well-known *Fekete-Szegő functional*, see [15, 21, 22]. The sharp upper bounds on $H_{2,2}(f)$ were obtained by the authors of articles [3, 18, 19, 23] for various classes of functions.

By the definition, $H_{3,1}(f)$ is given by

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

Note that for $f \in \mathcal{A}$, $a_1 = 1$ so that

$$H_{3,1}(f) = -a_2^2 a_5 + 2a_2 a_3 a_4 - a_3^3 + a_3 a_5 - a_4^2.$$

Obviously, the case of the upper bounds on $H_{3,1}(f)$ it is much more difficult than the cases of $H_{2,1}(f)$ and $H_{2,2}(f)$. In 2010, Babalola [2] has studied the $\max |H_{3,1}(f)|$ for the classes of starlike, convex and bounded turning functions.

Theorem A. *Let $f \in \mathcal{S}^*$, $h \in \mathcal{K}$ and $g \in \mathcal{R}$, respectively. Then*

$$|H_{3,1}(f)| \leq 16, \quad |H_{3,1}(h)| \leq \frac{32 + 33\sqrt{3}}{72\sqrt{3}} \approx 0.714,$$

and

$$|H_{3,1}(g)| \leq \frac{2736\sqrt{3} + 675\sqrt{5}}{4860\sqrt{3}} \approx 0.742.$$

Recently, Zaprawa [41] proved that

Theorem B. *Let $f \in \mathcal{S}^*$, $h \in \mathcal{K}$ and $g \in \mathcal{R}$, respectively. Then*

$$|H_{3,1}(f)| \leq 1, \quad |H_{3,1}(h)| \leq \frac{49}{540} \approx 0.090, \quad |H_{3,1}(g)| \leq \frac{41}{60} \approx 0.683.$$

Raza and Malik [37] have obtained the upper bound on $|H_{3,1}(f)|$ for a class of analytic functions that is related to the lemniscate of Bernoulli. Also, Bansal *et al.* [4] obtained the following results

Theorem C. *Let $h \in \mathcal{K}(-1/2)$ and $g \in \mathcal{R}$, respectively. Then*

$$|H_{3,1}(h)| \leq \frac{180 + 69\sqrt{15}}{32\sqrt{15}} \approx 3.609, \quad |H_{3,1}(g)| \leq \frac{439}{540} \approx 0.813.$$

For the class $\mathcal{R}(\alpha)$, Vamshee Krishna *et al.* [40] proved that

Theorem D. *Let $g \in \mathcal{R}(\alpha)$ with $0 \leq \alpha \leq \frac{1}{4}$. Then*

$$|H_{3,1}(g)| \leq \frac{(1-\alpha)^2}{3} \left[\frac{8(1-\alpha)}{9} + \frac{1}{4} \left(\frac{5-4\alpha}{3} \right)^{\frac{3}{2}} + \frac{4}{5} \right].$$

In the present investigation, our goal is to discuss the upper bounds to the third Hankel determinants for the subclasses of univalent functions: $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{R}(\alpha)$. Furthermore, we develop similar results on the Hankel determinants $|H_{3,1}(h)|$ and $|H_{3,1}(g)|$ in the context the close-to-convex harmonic mappings $f = h + \bar{g} \in \mathcal{M}(\alpha)$.

2 Preliminary results

Denote by \mathcal{P} the class of *Carathéodory functions* p normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad \text{and} \quad \Re(p(z)) > 0 \quad (z \in \mathbb{D}). \quad (2.1)$$

Following results are the well known for functions belonging to the class \mathcal{P} .

Lemma 1. [12] *If $p \in \mathcal{P}$ is of the form (2.1), then*

$$|p_n| \leq 2 \quad (n \in \mathbb{N}). \quad (2.2)$$

The inequality (2.2) is sharp and the equality holds for the function

$$\phi(z) = \frac{1+z}{1-z} = 1 + 2 \sum_{n=1}^{\infty} z^n.$$

Lemma 2. [27] *If $p \in \mathcal{P}$ is of the form (2.1), then holds the sharp estimate*

$$|p_n - p_k p_{n-k}| \leq 2 \quad (n, k \in \mathbb{N}, n > k). \quad (2.3)$$

Lemma 3. [16] *If $p \in \mathcal{P}$ is of the form (2.1), then holds the sharp estimate*

$$|p_n - \mu p_k p_{n-k}| \leq 2 \quad (n, k \in \mathbb{N}, n > k; 0 \leq \mu \leq 1). \quad (2.4)$$

Lemma 4. [25, 26] *If $p \in \mathcal{P}$ is of the form (2.1), then there exist x, z such that $|x| \leq 1$ and $|z| \leq 1$,*

$$2p_2 = p_1^2 + (4 - p_1^2)x, \quad (2.5)$$

and

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z. \quad (2.6)$$

3 Bounds of Hankel determinants for $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{R}(\alpha)$

In this section, we assume that

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{S}^*(\alpha), \quad h(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{K}(\alpha), \quad g(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{R}(\alpha).$$

Theorem 1. *Let $f \in \mathcal{S}^*(\alpha)$, $h \in \mathcal{K}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ with $0 \leq \alpha < 1$, respectively. Then*

$$|H_{3,1}(f)| \leq \frac{1}{18}(1-\alpha)^2(18-\alpha), \quad (3.1)$$

$$|H_{3,1}(h)| \leq \frac{1}{540}(1-\alpha)^2(49-16\alpha), \quad (3.2)$$

and

$$|H_{3,1}(g)| \leq \frac{1}{60}(1-\alpha)^2(36-20\alpha+5|1-4\alpha|). \quad (3.3)$$

Proof. Let

$$p(z) = \frac{1}{1-\alpha} \left(\frac{zf'(z)}{f(z)} - \alpha \right) \quad (0 \leq \alpha < 1; \quad z \in \mathbb{D}),$$

then, we have $\Re(p(z)) > 0$, and by elementary calculations, we obtain

$$p(z) = 1 + \frac{1}{1-\alpha}(a_2 z + (2a_3 - a_2^2)z^2 + \dots) = 1 + p_1 z + p_2 z^2 + \dots \quad (3.4)$$

It follows from (3.4) that

$$\begin{cases} a_2 &= (1-\alpha)p_1, \\ a_3 &= \frac{1}{2}(1-\alpha)[(1-\alpha)p_1^2 + p_2], \\ a_4 &= \frac{1}{6}(1-\alpha)[(1-\alpha)^2 p_1^3 + 3(1-\alpha)p_1 p_2 + 2p_3], \\ a_5 &= \frac{1}{24}(1-\alpha)[(1-\alpha)^3 p_1^4 + 6(1-\alpha)^2 p_1^2 p_2 + 8(1-\alpha)p_1 p_3 + 3(1-\alpha)p_2^2 + 6p_4]. \end{cases} \quad (3.5)$$

Hence, by using the above values of a_2 , a_3 , a_4 and a_5 from (3.5), and by a routine computation, we obtain

$$\begin{aligned} H_{3,1}(f) &= \frac{1}{144}(1-\alpha)^2 \left\{ -(1-\alpha)^4 p_1^6 + 3(1-\alpha)^3 p_1^4 p_2 + 8(1-\alpha)^2 p_1^3 p_3 - 9(1-\alpha)^2 p_1^2 p_2^2 \right. \\ &\quad \left. - 18(1-\alpha)p_1^2 p_4 + 24(1-\alpha)p_1 p_2 p_3 - 9(1-\alpha)p_2^3 + 18p_2 p_4 - 16p_3^2 \right\}. \end{aligned} \quad (3.6)$$

From (3.6), we have

$$\begin{aligned} H_{3,1}(f) &= \frac{1}{144}(1-\alpha)^2 \left\{ (1-\alpha)[p_2 - (1-\alpha)p_1^2]^3 - 16[p_3 - (1-\alpha)p_1 p_2]^2 \right. \\ &\quad \left. + 8[p_2 - (1-\alpha)p_1^2][p_4 - (1-\alpha)p_1 p_3] + 10[p_2 - (1-\alpha)p_1^2][p_4 - (1-\alpha)p_2^2] \right\}. \end{aligned}$$

We note that

$$0 < 1 - \alpha \leq 1 \quad \text{for} \quad 0 \leq \alpha < 1,$$

by triangle inequality and Lemma 3, we obtain the estimate (3.1) of $H_{3,1}(f)$.

Next, we consider $H_{3,1}(h)$. According to the Alexander relation, $b_k = ka_k$ ($k \in \mathbb{N}$). Putting it into the definition of $H_{3,1}(h)$ and applying the formula (3.5), we have

$$\begin{aligned} H_{3,1}(h) = \frac{1}{8640}(1-\alpha)^2 & \left\{ -(1-\alpha)^4 p_1^6 + 6(1-\alpha)^3 p_1^4 p_2 + 12(1-\alpha)^2 p_1^3 p_3 - 21(1-\alpha)^2 p_1^2 p_2^2 \right. \\ & \left. - 36(1-\alpha) p_1^2 p_4 + 36(1-\alpha) p_1 p_2 p_3 - 4(1-\alpha) p_2^3 + 72 p_2 p_4 - 60 p_3^2 \right\}. \end{aligned} \quad (3.7)$$

From (3.7), we have

$$\begin{aligned} H_{3,1}(h) = \frac{1}{8640}(1-\alpha)^2 & \left\{ 8(1-\alpha) \left[p_2 - \frac{1}{2}(1-\alpha) p_1^2 \right]^3 + 24 p_4 \left[p_2 - (1-\alpha) p_1^2 \right] \right. \\ & + 36 p_2 \left[p_4 - (1-\alpha) p_2^2 \right] + 12 \left[p_2 - (1-\alpha) p_1^2 \right] \left[p_4 - (1-\alpha) p_1 p_3 \right] \\ & \left. - 60 p_3 \left[p_3 - \frac{4}{5}(1-\alpha) p_1 p_2 \right] + 24(1-\alpha) p_2^2 \left[p_2 - \frac{3}{8}(1-\alpha) p_1^2 \right] \right\}. \end{aligned}$$

We observe that for $0 \leq \alpha < 1$ holds

$$1 - \alpha, \quad \frac{1}{2}(1 - \alpha), \quad \frac{4}{5}(1 - \alpha), \quad \frac{3}{8}(1 - \alpha) \in [0, 1].$$

By using Lemma 1 and Lemma 3 and triangle inequality, it easy to get the estimate (3.2) of $H_{3,1}(g)$.

Finally, for $H_{3,1}(g)$. Let

$$\frac{1}{1-\alpha} (g'(z) - \alpha) = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}.$$

If $g \in \mathcal{R}(\alpha)$, then

$$(k+1)c_{k+1} = (1-\alpha)p_k \quad (k \in \mathbb{N}). \quad (3.8)$$

Putting it into the definition of $H_{3,1}(g)$ and by the same way, we have

$$\begin{aligned} H_{3,1}(g) &= \frac{1}{2160}(1-\alpha)^2 \left\{ (1-\alpha) \left[-108 p_1^2 p_4 + 180 p_1 p_2 p_3 - 80 p_2^3 \right] + 144 p_2 p_4 - 135 p_3^2 \right\} \\ &= \frac{1}{2160}(1-\alpha)^2 \left\{ 108(1-\alpha) p_4 (p_2 - p_1^2) + 80(1-\alpha) p_2 (p_4 - p_2^2) \right. \\ &\quad \left. - 135 p_3 (p_3 - p_1 p_2) - 45(1-4\alpha) p_2 (p_4 - p_1 p_3) + (1+8\alpha) p_2 p_4 \right\}. \end{aligned}$$

Hence, it is easy to obtain the bound of $H_{3,1}(g)$. This completes the proof. \square

Remark 1. By setting $\alpha = 0$ in Theorem 1, we obtain the known results of Theorem B, and they are much better than Theorem A. Furthermore, the bounds of $H_{3,1}(g)$ in (3.3) improved and extended the result of the Theorem D.

In 1960, Lawrence Zalcman posed a conjecture that the coefficients of \mathcal{S} satisfy the sharp inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2 \quad (n \in \mathbb{N}),$$

with equality only for the Koebe function $k(z) = z/(1-z)^2$ and its rotations. We call $J_n(f) = a_n^2 - a_{2n-1}$ the Zalcman functional for $f \in \mathcal{S}$.

We observe that $H_{3,1}(f)$ ($f \in \mathcal{A}$) can be written in the form

$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) - a_5J_2(f),$$

and equivalently,

$$H_{3,1}(f) = a_3J_3(f) + a_4(2a_2a_3 - a_4) - a_5a_2^2.$$

An analogous calculation can be applied to the Zalcman functional $J_n(f)$ for the classes of starlike, convex and bounded turning functions of order α .

Theorem 2. *The following estimates hold for $J_n(f)$:*

1. If $f \in \mathcal{S}^*(\alpha)$ ($0 \leq \alpha < 1$), then $J_3(f) \leq \frac{1}{2}(1-\alpha)(8-7\alpha)$.
2. If $h \in \mathcal{K}(\alpha)$ ($-1/2 \leq \alpha < 1$), then $J_3(h) \leq \frac{1}{360}(1-\alpha)(127-109\alpha)$.
3. If $g \in \mathcal{R}(\alpha)$ ($0 \leq \alpha < 1$), then $J_n(g) \leq \frac{2}{2n-1}(1-\alpha) \quad (n \geq 2)$.

Proof. Let $f \in \mathcal{S}^*(\alpha)$, from (3.5), it follow that

$$\begin{aligned} J_3(f) &= \frac{1}{24}(1-\alpha) \left\{ -5(1-\alpha)^3p_1^4 - 6(1-\alpha)^2p_1^2p_2 - 3(1-\alpha)p_2^2 + 8(1-\alpha)p_1p_3 + 6p_4 \right\} \\ &= \frac{1}{24}(1-\alpha) \left\{ -5(1-\alpha)[p_2 - (1-\alpha)p_1^2]^2 + 8(1-\alpha)p_1[p_3 - (1-\alpha)p_1p_2] \right. \\ &\quad \left. + 8(1-\alpha)p_2[p_2 - (1-\alpha)p_1^2] + 6[p_4 - (1-\alpha)p_2^2] \right\}. \end{aligned}$$

By using Lemma 1 and Lemma 3, we obtain the above bound for the Zalcman functional $J_3(f)$.

Combining the Alexander relation, $b_k = ka_k$, and the formula (3.5), yields

$$\begin{aligned} J_3(h) &= \frac{1}{360}(1-\alpha) \left\{ -7(1-\alpha)^3p_1^4 - 2(1-\alpha)^2p_1^2p_2 - (1-\alpha)p_2^2 + 24(1-\alpha)p_1p_3 + 18p_4 \right\} \\ &= \frac{1}{360}(1-\alpha) \left\{ -\frac{63}{4}(1-\alpha)[p_2 - \frac{2}{3}(1-\alpha)p_1^2]^2 + 24(1-\alpha)p_1[p_3 - \frac{2}{3}(1-\alpha)p_1p_2] \right. \\ &\quad \left. + \frac{21}{2}(1-\alpha)p_2[p_2 - \frac{2}{3}(1-\alpha)p_1^2] + \frac{17}{4}(1-\alpha)p_2^2 + 18p_4 \right\}. \end{aligned}$$

Again, by using Lemma 1 and Lemma 3, we obtain the bound for the Zalcman functional $J_3(h)$.

For $g \in \mathcal{R}(\alpha)$, according to the formula (3.8), we have

$$\begin{aligned} J_n(g) &= \frac{1}{n^2}(1-\alpha)^2p_{n-1}^2 - \frac{1}{2n-1}(1-\alpha)p_{2n-2} \\ &= -\frac{1}{2n-1}(1-\alpha) \left[p_{2n-2} - \frac{2n-1}{n^2}(1-\alpha)p_{n-1}^2 \right]. \end{aligned}$$

In view of

$$0 < \frac{2n-1}{n^2}(1-\alpha) < 1 \quad (0 \leq \alpha < 1; n \geq 2),$$

and, by Lemma 3, we have the desired bound of the Zalcman functional $J_n(g)$. This completes the proof. \square

Remark 2. By setting $\alpha = -1/2$ for the class $\mathcal{K}(\alpha)$ in Theorem 2, we obtain the known results [1, Theorem 2.3]. Furthermore, using the similar argument in Theorem 2, we may obtain the bounds of the Zalcman functional $J_2(f)$ and $J_2(h)$: If $f \in \mathcal{S}^*(\alpha)$ ($0 \leq \alpha < 1$), then $J_2(f) \leq 1 - \alpha$. If $h \in \mathcal{K}(\alpha)$ ($-1/2 \leq \alpha < 1$), then $J_2(h) \leq \frac{1}{3}(1 - \alpha)$.

4 Bounds of Hankel determinants for $\mathcal{M}(\alpha)$

In this section, we obtain upper bounds for the Hankel determinants $|H_{3,1}(h)|$ and $|H_{3,1}(g)|$ of close-to-convex harmonic mappings $f = h + \overline{g} \in \mathcal{M}(\alpha)$.

Theorem 3. Let $f = h + \overline{g} \in \mathcal{M}(\alpha)$ be of the form (1.2). Then

$$|H_{3,1}(h)| \leq \frac{1}{540}(1-\alpha)^2(15\alpha^2 - 34\alpha + 52),$$

and

$$|H_{3,1}(g)| \leq \frac{1}{30}(1-\alpha).$$

Proof. Let

$$p(z) = \frac{1}{1-\alpha} \left(1 + \frac{zh''(z)}{h'(z)} - \alpha \right) = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P} \quad \left(-\frac{1}{2} \leq \alpha < 1; z \in \mathbb{D} \right).$$

Using the same method of Theorem 1, we get the expression of $H_{3,1}(h)$ is the formula (3.7). We give another decomposition for functional $H_{3,1}(h)$ as follows

$$\begin{aligned} H_{3,1}(h) = \frac{1}{8640}(1-\alpha)^2 & \left\{ 8(1-\alpha) \left[p_2 - \frac{1}{2}(1-\alpha)p_1^2 \right]^3 - 60 \left[p_3 - \frac{1}{2}(1-\alpha)p_1p_2 \right]^2 \right. \\ & + 48 \left[p_2 - \frac{1}{2}(1-\alpha)p_1^2 \right] \left[p_4 - \frac{1}{2}(1-\alpha)p_1p_3 \right] - 15(1-\alpha)^2 p_1^2 p_2^2 \\ & \left. + 24 \left[p_2 - \frac{1}{2}(1-\alpha)p_1^2 \right] \left[p_4 - \frac{1}{2}(1-\alpha)p_2^2 \right] \right\}. \end{aligned}$$

We note that

$$0 \leq \frac{1}{2}(1-\alpha) \leq 1 \quad \text{for } 0 \leq \alpha < 1,$$

by triangle inequality and Lemmas 1-3, we can obtain the estimate of $H_{3,1}(h)$.

By the power series representations of h and g for $f = h + \overline{g} \in \mathcal{M}(\alpha)$, we see that

$$b_1 = 0, \quad (k+1)b_{k+1} = ka_k \quad \text{for } k \geq 1,$$

which yields

$$\begin{cases} b_2 = \frac{1}{2}a_1 = \frac{1}{2}, \\ b_3 = \frac{2}{3}a_2 = \frac{1}{3}(1-\alpha)p_1, \\ b_4 = \frac{3}{4}a_3 = \frac{1}{8}[(1-\alpha)^2p_1^2 + (1-\alpha)p_2], \\ b_5 = \frac{4}{5}a_4 = \frac{1}{30}[(1-\alpha)^3p_1^3 + 3(1-\alpha)^2p_1p_2 + 2(1-\alpha)p_3]. \end{cases}$$

Then, by using (2.5) and (2.6) in Lemma 4, we obtain that for some x and z such that $|x| \leq 1$ and $|z| \leq 1$,

$$\begin{aligned} H_{3,1}(g) &= 2b_2b_3b_4 - b_3^3 - b_2^2b_5 = b_3b_4 - b_3^3 - \frac{1}{4}b_5 \\ &= \frac{1}{2160}(1-\alpha) \left\{ (-8\alpha^2 + 16\alpha + 1)p_1^3 + 9(4 - p_1^2)[p_1x^2 - 2(1 - |x|^2)z] \right\}. \end{aligned}$$

By Lemma 1, we may assume that $|p_1| = c \in [0, 2]$. By applying the triangle inequality in above relation with $\mu = |x|$, we obtain

$$|H_{3,1}(g)| \leq \frac{1}{2160}(1-\alpha) \left\{ |8\alpha^2 - 16\alpha - 1|c^3 + 9(4 - c^2)[(c-2)\mu^2 + 2] \right\} =: Q(c, \mu).$$

We note that

$$(c-2)\mu^2 + 2 \leq 2, \quad \text{for } \mu \in [0, 1] \quad \text{and} \quad c \in [0, 2].$$

Hence, we have

$$|H_{3,1}(g)| \leq Q(c, \mu) \leq Q(c, 0) = \frac{1}{2160}(1-\alpha) \left\{ |8\alpha^2 - 16\alpha - 1|c^3 - 18c^2 + 72 \right\}.$$

Let

$$\chi(c) = |8\alpha^2 - 16\alpha - 1|c^3 - 18c^2 + 72 \quad (c \in [0, 2]).$$

Then, we obtain

$$\chi'(c) = 3c(|8\alpha^2 - 16\alpha - 1|c - 12),$$

and

$$\chi''(c) = 6(|8\alpha^2 - 16\alpha - 1|c - 6).$$

Solving the equation $\chi'(c) = 0$, we get the critical points are $c = 0$ and

$$C_1 = \frac{12}{|8\alpha^2 - 16\alpha - 1|}.$$

We observe that

$$\chi''(c)\Big|_{c=0} = -36 < 0, \quad \chi''(c)\Big|_{c=C_1} = 36 > 0,$$

and

$$0 \leq |8\alpha^2 - 16\alpha - 1| \leq 9 \quad (-1/2 \leq \alpha < 1).$$

Hence, we get

$$\chi(c) \leq \max \left\{ \chi(0), \chi(2) \right\} = \max \left\{ 72, 8|8\alpha^2 - 16\alpha - 1| \right\} = 72.$$

Thus, we obtain the following bound

$$|H_{3,1}(g)| \leq \frac{1}{30}(1 - \alpha).$$

□

Remark 3. In order to obtain the bounds of $H_{3,1}(h)$, we give two kinds of decomposition for formula (3.7) in Theorem 1 and Theorem 3, respectively. Hence, it is a natural question: Whether there is an optimal decomposition for the similar formulae.

Remark 4. For $H_{3,1}(g)$ in Theorem 3, if we apply the method in Theorem 1, then

$$\begin{aligned} H_{3,1}(g) &= 2b_2b_3b_4 - b_3^3 - b_2^2b_5 = b_3b_4 - b_3^3 - \frac{1}{4}b_5 \\ &= \frac{1}{540}(1 - \alpha) \left\{ -2(1 - \alpha)^2p_1^3 - 9[p_3 - (1 - \alpha)p_1p_2] \right\} \\ &= \frac{1}{540}(1 - \alpha) \left\{ 3(1 - \alpha)p_1[p_2 - \frac{2}{3}(1 - \alpha)p_1^2] - 9[p_3 - \frac{2}{3}(1 - \alpha)p_1p_2] \right\}. \end{aligned}$$

By using Lemmas 1 and 3, we have

$$|H_{3,1}(g)| \leq \frac{1}{90}(1 - \alpha)(5 - 2\alpha),$$

obviously,

$$\frac{1}{90}(1 - \alpha)(5 - 2\alpha) > \frac{1}{30}(1 - \alpha) \quad \text{for} \quad -\frac{1}{2} \leq \alpha < 1.$$

Hence, we choose the bound of in $H_{3,1}(g)$ in Theorem 3.

Corollary 1. Let $f = h + \overline{g} \in \mathcal{M}(-1/2)$ be of the form (1.2). Then

$$|H_{3,1}(h)| \leq \frac{291}{960} \approx 0.303125, \quad |H_{3,1}(g)| \leq \frac{1}{20} = 0.05.$$

Remark 5. The result of $H_{3,1}(h)$ in Corollary 1 is much better than Theorem C (see [4, Theorem 2.7]). From the upper bounds of $H_{3,1}(h)$ and $H_{3,1}(g)$, we note that the former is much larger than the latter, this implies that the analytic part h accounts for absolute advantage than the co-analytic part g for the harmonic mappings $f = h + \overline{g} \in \mathcal{M}(\alpha)$.

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